Geomag and Rigidity

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Geomag is a toy that, in its simplest form, consists of steel ball bearings and magnetic rods that can be put together to create a wide range of objects. For example, Figure 1 shows a Geomag model of a regular icosahedron.



Figure 1: A Geomag model of a regular icosahedron

That model is rigid, it does not deform if you try moving the ball bearings that are the vertices of the model, or the rods that are the edges of the model. But take away even a single rod and the shape will easily deform – moving the two formerly connected ball bearings towards or away from each other.

Why? It is a well-known result that such shapes – considering only shapes that are an external shell, with no internal bracing – are rigid if made up entirely of triangles. The architectural works of Buckminster Fuller, see for example Figure 2, are one application of that.



Figure 2: The Montreal Biosphere by Buckminster Fuller

Image by Cédric Thévenet
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https://commons.wikimedia.org/wiki/File:Biosphère_Montréal.jpg

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But we can do better than well-known. We consider only convex shapes – or near-convex shapes as we are limited by that all Geomag rods have the same length. These satisfy the well-known Euler equation:

$$F + V = E + 2 \qquad (1)$$

where V is the number of vertices, E is the number of edges and F is the number of faces. There is nothing physically in the model representing the faces, but we can see them if we remember that the shape is convex.

We now need two more things. The first is that we assume that the mean number of edges around each face is γ. With an all triangle model $\gamma = 3$, otherwise $\gamma > 3$. Allowing for that each edge belongs to two faces:

$$2E = \gamma F \qquad (2)$$

Now consider each ball bearing. We can put it anywhere in three dimensional space, so the number of degrees of freedom we have is $3V$. But each rod we use fixes the distance between two ball bearings, removing one degree of freedom. The total number of degrees of freedom we have, d, is thus given by:

$$d = 3V - E \qquad (3)$$

We can rewrite (1) as:

$$V = E - F + 2 \qquad (4)$$

and substitute that into (3) as:

$$d = 2E - 3F + 6 \qquad (5)$$

and then using (2) in (5):

$$d = (\gamma - 3)F + 6 \qquad (6)$$

In this final term, 6 is the number of degrees of freedom of a rigid body: 3 translational and 3 rotational. This leaves $(\gamma - 3)F$ as the number of degrees of deformational freedom, which is, as we expected, zero if and only if $\gamma = 3$, i.e. all faces are triangles. Note that from (3) each edge we remove allows one degree of deformational freedom, hence the observed behaviour that removing one rod from the model allows the two formerly connected ball bearings to be moved towards and away from each other.

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A further example is shown in Figure 3:

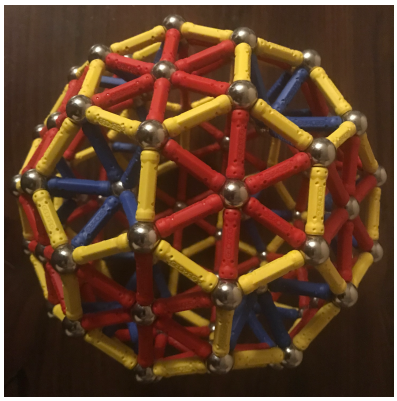


Figure 3: A Geomag model of a buckyball (truncated icosahedron)

The yellow (lighter coloured) rods are a model of a buckyball, named after Buckminster Fuller, or formally a truncated icosahedron, most familiar as a football. It does not consist of just triangles and is far from rigid – in fact it collapses under its own weight without the other (red and blue) rods. But those added rods make it entirely out of triangles and, although not strictly convex as the blue (darkest) rods are too long and are indented, it also is rigid, as can be seen when built. Also rigid is the figure shown in Figure 4:

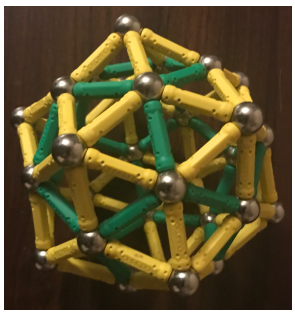


Figure 4: A Geomag model of the dual of a buckyball (pentakis dodecahedron)

This – using both colours of rod – is a model of the dual of a buckyball, or formally a pentakis dodecahedron, where the faces and vertices of the buckyball are exchanged. The yellow (lighter) rods are slightly too long for the shape to be convex, but that does not affect its rigidity.

We can consider such rigid models, with $2E = 3F$, further. In most such models, each vertex is joined by one of 3, 4, 5 or 6 edges, and each is at a corner of the same number of faces. Let the numbers of such vertices be V_3, V_4, V_5 and V_6. Then:

$$V = V_3 + V_4 + V_5 + V_6 \qquad (7)$$

and:

$$2E = 3V_3 + 4V_4 + 5V_5 + 6V_6 \qquad (8)$$

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and hence also:

$$3F = 3V_3 + 4V_4 + 5V_5 + 6V_6 \qquad (9)$$

Multiplying Euler's equation (1) by 6, substituting in (7), (8) and (9) to replace V, E and F, and then cancelling terms, leaves:

$$3V_3 + 2V_4 + V_5 = 12 \qquad (10)$$

Note that V_6 has disappeared from this equation. Examples include a tetrahedron, with just $V_3 = 4$, and an octahedron, with just $V_4 = 6$. In many such polyhedra, including the dual of the buckyball, there are only vertices with 5 or 6 edges, and faces joining each of them, so $V_3 = V_4 = 0$, and hence $V_5 = 12$. The limiting case is the icosahedron with $V_6 = 0$, the dual of the buckyball has $V_6 = 20$.

An example of this can be seen in Figure 5. If this near-sphere were complete it would have exactly 12 vertices where 5 faces and 5 edges meet. An example of such a vertex can be seen in this picture, slightly below the point midway between the two reflected clouds.



Figure 5: La Géode at La Cité des Sciences et de l'Industrie, Paris
by Adrien Fainsilber and Gérard Chamayou

From picture by DiscoA340 - Own work, CC BY-SA 4.0, https://en.wikipedia.org/w/index.php?curid=75518766

Another application of this result is to duals of these polyhedra, such as the buckyball (as the dual of the dual of itself), where with only pentagon and hexagon faces then there must be exactly 12 pentagonal faces, as is the case for the model in Figure 3, considering only the yellow (lighter) rods. The limiting case of this is the dodecahedron, with 12 pentagonal faces and no hexagonal faces.

Disclosure

The author reports there are no competing interests to declare.

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